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# On discretization of tori of compact simple Lie groups

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Received 25 March 2009

Published 7 September 2009

Online at [stacks.iop.org/JPhysA/42/385208](http://stacks.iop.org/JPhysA/42/385208)

## Abstract

Three types of numerical data are provided for simple Lie groups of any type and rank. These data are indispensable for Fourier-like expansions of multidimensional digital data into finite series of  $C$ - or  $S$ -functions on the fundamental domain  $F$  of the underlying Lie group  $G$ . Firstly, we determine the number  $|F_M|$  of points in  $F$  from the lattice  $P_M^\vee$ , which is the refinement of the dual weight lattice  $P^\vee$  of  $G$  by a positive integer  $M$ . Secondly, we find the lowest set  $\Lambda_M$  of dominant weights, specifying the maximal set of  $C$ - and  $S$ -functions that are pairwise orthogonal on the point set  $F_M$ . Finally, we describe an efficient algorithm for finding, on the maximal torus of  $G$ , the number of conjugate points to every point of  $F_M$ . Discrete  $C$ - and  $S$ -transforms, together with their continuous interpolations, are presented in full generality.

PACS numbers: 02.20.-a, 02.30.Gp, 02.30.Nw, 02.60.Ed

## 1. Introduction

Processing of multidimensional digital data is the general motivation for this paper. However, our immediate objective is much more specific: recently introduced new Fourier-like transforms, called  $C$ -,  $S$ - and  $E$ -transforms [1], or simply orbit function transforms, on lattices of any dimension, symmetry and density, reviewed in [2–4] (see also references therein), require certain technical data to be truly versatile and useful tools in all cases. Providing such data in full generality is the main goal of this paper. For the lowest dimension,  $n = 2$  and  $3$ , it was possible to obtain the data by simpler means, although even then it presented a challenge of sorts [5–9].

The theoretical background of this paper is the amazing uniformity, as to the type and rank, of the theory of simple Lie algebras over the complex number field [10–12] and related finite Weyl groups [13, 14], also referred to as crystallographic Coxeter groups. Discretization

problems frequently call for the properties of the maximal torus of the underlying compact simple Lie group [15].

Data provided in the paper can be used in many investigations. The focus is on three types of transforms defined on a well-known finite region  $F$  of a real Euclidean space of dimension  $n$ , where  $n$  is the rank of the underlying simple Lie group  $G$ . The region for digital data is the lattice fragment  $F_M$ , which is the intersection of  $F$  with the dual weight lattice of  $G$ , refined by the factor  $1/M$ . Positive integer  $M$  fixes the density of points in  $F_M$ . The symmetry of the dual weight lattice is dictated by our choice of  $G$ . Digital data are functions sampled on  $F_M$ . They may be of wildly varying densities. Therefore, the flexibility in the choice of  $M$  is crucial in our set-up.

The results in this paper provide all the information that is needed for three types of expansions of functions, given on the fundamental region  $F$  of any compact simple Lie group/Lie algebra. The fundamental region is an  $n$ -dimensional simplex given explicitly by its  $n + 1$  vertices. When the Lie group/Lie algebra is semisimple but not simple, the data needed are a straightforward concatenation of those needed for simple cases. In particular, the fundamental region is the Cartesian product of simplexes of simple constituents of the Lie group/Lie algebra.

Firstly, we consider the expansion into series of either  $C$ - or  $S$ -functions, analog of the expansions into Fourier series in  $n$  variables. Unlike the familiar case, our variables do not refer necessarily to an orthogonal basis in  $\mathbb{R}^n$ . That is just one of the cases when the underlying Lie algebra is the product  $A_1 \times A_1 \times \cdots \times A_1$  of  $n$  copies of  $A_1$ . Here the variables are defined relative to a basis which reflects the symmetries of the weight lattice of the Lie algebra. Hence, in all but one case, the basis is non-orthogonal.

Secondly, we consider the expansion of functions sampled on a lattice grid  $F_M \subset F$  with density controlled by  $M \in \mathbb{N}$ . Such expansions are necessarily finite because the number of points  $|F_M|$  is finite. The numbers  $|F_M|$  are determined here for all simple Lie algebras and for all  $M$ .

Our third case examines the continuous interpolations of functions sampled on  $F_M$ . Once the corresponding finite series is found, it suffices to replace the lattice variables in the expansion functions by the continuous variables. Such a simple idea applies equally well to any dimension, to lattices of any symmetry and to grids in  $F$  of any density. The interpolation is performed in the ‘Fourier space’ rather than in the space of the data, where the best of the common interpolation techniques are usually applied. At present there is some empirical evidence that the quality of our interpolation method compares favorably with the best methods used in 2D, while being as fast as the simplest of the standard interpolations in 2D, namely the linear interpolation. In dimensions exceeding 2, there is hardly any competition to the possibilities brought forth here.

The technical data provided in this paper in full generality are as follows. First we provide the formula for the volume  $\text{vol}(F)$  of the fundamental region of  $G$ , which is closely related to the order of the Weyl group  $W$  of  $G$ .

A new result presented in this paper is the number  $|F_M|$  of lattice points in the fragment  $F_M$  for all  $1 \leq M < \infty$  and for all simple Lie groups  $G$ . The number  $|F_M|$  is used in the discrete orthogonality relation of the orbit functions when those are sampled at the points of  $F_M$ .

The second result presented in this paper is the determination of the sets  $\Lambda_M$  of  $C$ - and  $S$ -functions which are pairwise orthogonal on the point sets  $F_M$ . Such a set consists of points of the weight lattice  $P$  of  $G$ . In fact there is an unlimited number of such sets in  $P$ . Our aim is to describe the unique lowest set.

The third result presented in this paper is a method for finding a number indispensable to the definition of the inner product of functions sampled on the points of  $F_M$ . The number is equal to the number of elements in a conjugacy class of elements of finite order on the maximal torus of a simple Lie group. That number, and also an implicit method for determining it, has already been found in [16]. The present prescription is explicit and easy to use in all cases.

In order to determine the numbers  $|F_M|$  and the sets  $\Lambda_M$ , we must review the orthogonality properties of orbit functions in a somewhat wider context than in [16, 17].

Reasons for which the results presented should be of interest, beyond our own motivations, are as follows.

The volume of  $F$  for each  $G$  is needed when the continuous orthogonality of orbit functions is used. Its relation to the order of the Weyl group is of independent interest.

The points of  $F_M$  are obtained in terms of barycentric coordinates  $[s_0, s_1, \dots, s_n]$  satisfying a certain sum rule (19). The subset of  $F_M$ , subject to the additional requirement that  $\gcd\{s_0, s_1, \dots, s_n\} = 1$ , consists of the representatives of the conjugacy classes of elements of adjoint order  $M$  in the corresponding compact simple Lie group. Counting such conjugacy classes in certain Lie groups was the subject of the papers [18–20].

The sets  $\Lambda_M$  and  $F_M$  are in one-to-one correspondence. Most of the properties of elements of  $F_M$  are therefore reflected in the corresponding elements of  $\Lambda_M$ .

The method used here to determine the number of conjugate points to every point of  $F_M$  that can be extracted from much more elaborate techniques exploited in [21] for describing the faces of all dimensions of the proximity (Voronoi) cells of the root lattices of  $G$ .

The pertinent standard properties of simple Lie algebras are recalled in section 2. A general formula for the volume of  $F$  is given there. In section 3 the lattice grids  $F_M$  and the set  $\Lambda_M$  are described and the number of their points is found. Section 4 contains a description of the  $C$ - and  $S$ -functions and their properties pertinent for our goals. In view of the preceding results, section 5 is devoted to the detailed description of the discrete orthogonality of  $C$ - and  $S$ -functions. The distinction between the two is brought out. Comments and interesting follow-up questions are in the last section.

## 2. Pertinent properties of Lie groups and Lie algebras

### 2.1. Definitions and notations

Consider the Lie algebra of the compact simple Lie group  $G$  of rank  $n$ , with the set of simple roots  $\Delta = \{\alpha_1, \dots, \alpha_n\}$ , spanning the Euclidean space  $\mathbb{R}^n$ .

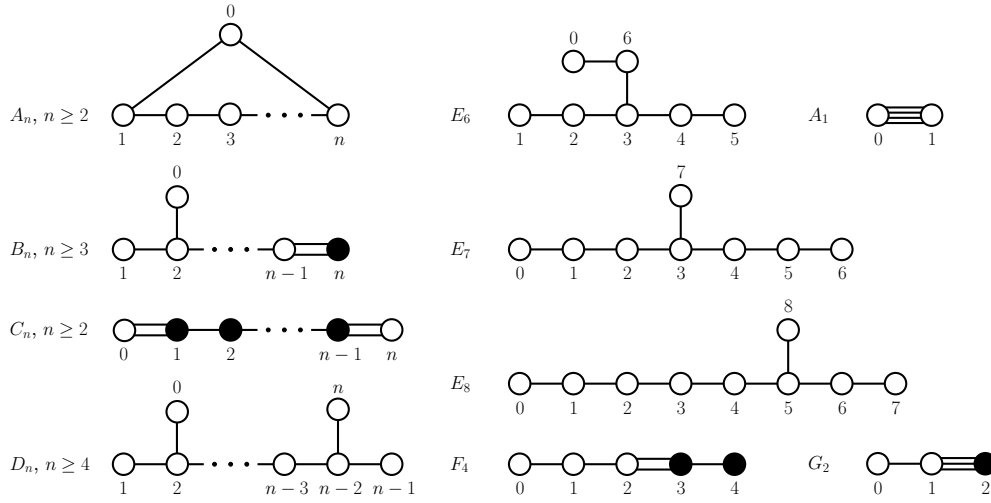
By uniform and standard methods for  $G$  of any type and rank, one determines from  $\Delta$  a number of related quantities and virtually all the properties of  $G$ . We make use of the following ones.

- The highest root  $\xi \equiv -\alpha_0 = m_1\alpha_1 + \dots + m_n\alpha_n$ . Here the coefficients  $m_j$  are known positive integers also called the marks of  $G$ .
- The Coxeter number  $m = 1 + m_1 + \dots + m_n$  of  $G$ .
- The Cartan matrix  $C$

$$C_{ij} = \frac{2\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_j, \alpha_j\rangle}, \quad i, j \in \{1, \dots, n\}.$$

- The order  $c$  of the center of  $G$ :

$$c = \det C. \tag{1}$$



**Figure 1.** The extended Coxeter–Dynkin diagrams of simple Lie algebras and their numbering. The extension root carries number 0.

- The root lattice  $Q$  of  $G$ :

$$Q = \{ \alpha \in \mathbb{R}^n \mid \langle \alpha, \omega^\vee \rangle \in \mathbb{Z}, \forall \omega^\vee \in P^\vee \} = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n. \quad (2)$$

- The  $\mathbb{Z}$ -dual lattice to  $Q$ :

$$P^\vee = \{ \omega^\vee \in \mathbb{R}^n \mid \langle \omega^\vee, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta \} = \mathbb{Z}\omega_1^\vee + \dots + \mathbb{Z}\omega_n^\vee.$$

- The dual root lattice

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee, \quad \text{where } \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}.$$

- The extended Coxeter–Dynkin diagram (DD) of  $G$ —this diagram describes the system of the simple roots  $\Delta$  together with the highest root  $\xi = -\alpha_0$ . The  $k$ th node corresponds to the vector  $\alpha_k, k = 0, \dots, n$ . Direct links between two nodes indicate the absence of orthogonality between the pair of the corresponding vectors. Single, double and triple vertices imply that relative angles between these vectors are  $2\pi/3, 3\pi/4, 5\pi/6$ , respectively. Quadruple vertex, which appears for the case  $A_1$  only, denotes the fact that the highest root and the simple root coincide. Colors of the nodes indicate relative length of  $\alpha_k$ . For the cases  $B_n, C_n$  and  $F_4$  the squared length of the black node (short root) is half of the squared length of the white node (long root). For  $G_2$ , the squared length of the black node is one third of the squared length of the white node. We also use the standard additional convention for the squared lengths of the white nodes  $\alpha$ :

$$\langle \alpha, \alpha \rangle = 2.$$

The original (non-extended) Coxeter–Dynkin diagram of  $G$  can be recovered from extended DD by omitting the extension 0-node and adjacent edges.

The extended DDs of all simple Lie algebras are shown in figure 1.

### 2.2. Weyl group and affine Weyl group

The properties of Weyl groups and affine Weyl groups can be found for example in [13, 14]. The finite Weyl group  $W$  is generated by  $n$  reflections  $r_\alpha, \alpha \in \Delta$ , in  $(n - 1)$ -dimensional ‘mirrors’ orthogonal to simple roots intersecting at the origin:

$$r_{\alpha_i} a \equiv r_i a = a - \frac{2\langle a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \quad a \in \mathbb{R}^n.$$

The infinite affine Weyl group  $W^{\text{aff}}$  is generated by translations by elements of  $Q^\vee$  and by  $W$ . More precisely,  $W^{\text{aff}}$  is the semidirect product of the Abelian group of translations  $Q^\vee$  and of the Weyl group  $W$ :

$$W^{\text{aff}} = Q^\vee \rtimes W. \tag{3}$$

Equivalently,  $W^{\text{aff}}$  is generated by the reflections  $r_i$  and reflection  $r_0$ , where

$$r_0 a = r_\xi a + \frac{2\xi}{\langle \xi, \xi \rangle}, \quad r_\xi a = a - \frac{2\langle a, \xi \rangle}{\langle \xi, \xi \rangle} \xi, \quad a \in \mathbb{R}^n.$$

The fundamental region  $F$  of  $W^{\text{aff}}$  is the convex hull of the points  $\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\}$ :

$$F = \{y_1 \omega_1^\vee + \dots + y_n \omega_n^\vee \mid y_0, \dots, y_n \in \mathbb{R}^{\geq 0}, y_0 + y_1 m_1 + \dots + y_n m_n = 1\}$$

$$= \{a \in \mathbb{R}^n \mid \langle a, \alpha \rangle \geq 0, \forall \alpha \in \Delta, \langle a, \xi \rangle \leq 1\} \tag{4}$$

and

$$W^{\text{aff}} F = \mathbb{R}^n.$$

Since  $F$  is a fundamental region of  $W^{\text{aff}}$ , we have the following.

- (i) For any  $a \in \mathbb{R}^n$  there exist  $a' \in F, w \in W$  and  $q^\vee \in Q^\vee$  such that

$$a = wa' + q^\vee. \tag{5}$$

- (ii) If  $a, a' \in F$  and  $a' = w^{\text{aff}} a, w^{\text{aff}} \in W^{\text{aff}}$ , then  $a = a'$ , i.e. if there exist  $w \in W$  and  $q^\vee \in Q^\vee$  such that  $a' = wa + q^\vee$ , then

$$a' = a = wa + q^\vee. \tag{6}$$

- (iii) Consider a point  $a = y_1 \omega_1^\vee + \dots + y_n \omega_n^\vee \in F$ , such that  $y_0 + y_1 m_1 + \dots + y_n m_n = 1$ . The isotropy group

$$\text{Stab}_{W^{\text{aff}}}(a) = \{w^{\text{aff}} \in W^{\text{aff}} \mid w^{\text{aff}} a = a\} \tag{7}$$

of the point  $a$  is trivial,  $\text{Stab}_{W^{\text{aff}}}(a) = 1$ , if  $a \in \text{int}(F)$ , where  $\text{int}(F)$  denotes the interior of  $F$ , i.e. all  $y_i > 0, i = 0, \dots, n$ . Otherwise the group  $\text{Stab}_{W^{\text{aff}}}(a)$  is generated by such  $r_i$  for which  $y_i = 0, i = 0, \dots, n$ . The Coxeter–Dynkin diagram of  $\text{Stab}_{W^{\text{aff}}}(a)$  is obtained as a subgraph of the extended DD by omitting such vertices  $\alpha_i$  (and adjacent edges) for which  $y_i > 0, i = 0, \dots, n$ . Since there always exists some  $y_i > 0, i = 0, \dots, n$ , this subgraph is always proper.

### 2.3. Volume of $F$

The volumes of fundamental domains  $\text{vol}(F)$  often appear in an implicit form in the literature and are commonly used for the calculation of the order of the Weyl group [10, 13]. For applications, it may be also useful to calculate explicit values of these volumes which we

summarize in table 2. Taking into account the standard convention for the lengths of the root vectors, one can use for the calculation of  $\text{vol}(F)$  the following formula.

**Proposition 2.1.** *The volume  $\text{vol}(F)$  of the fundamental domain  $F \subset \mathbb{R}^n$  is equal to*

$$\text{vol}(F) = \frac{1}{n!} \frac{1}{m_1 \cdots m_n} \sqrt{\frac{2}{\langle \alpha_1, \alpha_1 \rangle} \cdots \frac{2}{\langle \alpha_n, \alpha_n \rangle}} c^{-\frac{1}{2}}.$$

**Proof.** Suppose we have standard orthonormal basis  $\chi$  of  $\mathbb{R}^n$ . The map which transforms standard  $n$ -simplex into  $F$  is determined by the matrix  $\omega_m^\vee$  given by the coordinates of the vectors  $\omega_i^\vee/m_i, i \in \{1, \dots, n\}$ , in the basis  $\chi$ . Since the volume of the standard  $n$ -simplex is  $1/n!$ , we have

$$\text{vol}(F) = \frac{1}{n!} |\det \omega_m^\vee|.$$

Let  $\omega^\vee, \alpha^\vee$  denote the matrices given by the coordinates (in columns) of the vectors  $\omega_i^\vee$  and  $\alpha_i^\vee$ , respectively and let  $G(\alpha_1^\vee, \dots, \alpha_n^\vee)$  denote the Gram matrix of the basis  $\alpha_1^\vee, \dots, \alpha_n^\vee$ , i.e.  $G(\alpha_1^\vee, \dots, \alpha_n^\vee)_{ij} = \langle \alpha_i^\vee, \alpha_j^\vee \rangle$ . Then  $(\alpha^\vee)^T \alpha^\vee = G(\alpha_1^\vee, \dots, \alpha_n^\vee)$  and  $c \det \omega^\vee = \det \alpha^\vee$ . Clearly, the relation

$$\det G(\alpha_1^\vee, \dots, \alpha_n^\vee) = \frac{2}{\langle \alpha_1, \alpha_1 \rangle} \cdots \frac{2}{\langle \alpha_n, \alpha_n \rangle} c$$

also holds. We now have

$$\begin{aligned} |\det \omega_m^\vee| &= \frac{1}{m_1 \cdots m_n} |\det \omega^\vee| = \frac{1}{m_1 \cdots m_n} c^{-1} |\det \alpha^\vee| \\ &= \frac{1}{m_1 \cdots m_n} c^{-1} \sqrt{\det G(\alpha_1^\vee, \dots, \alpha_n^\vee)}. \end{aligned}$$

□

#### 2.4. Action of $W$ on the maximal torus $\mathbb{R}^n/Q^\vee$

If we have two elements  $a, a' \in \mathbb{R}^n$  such that  $a' - a = q^\vee$ , with  $q^\vee \in Q^\vee$ , then for  $w \in W$  we have  $wa - wa' = wq^\vee \in Q^\vee$ , i.e. we have a natural action of  $W$  on the torus  $\mathbb{R}^n/Q^\vee$ . For  $x \in \mathbb{R}^n/Q^\vee$  we denote the isotropy group

$$\text{Stab}(x) = \{w \in W \mid wx = x\}$$

and its order by  $h_x \equiv |\text{Stab}(x)|$ . We denote the orbit

$$Wx = \{wx \in \mathbb{R}^n/Q^\vee \mid w \in W\}$$

and its order by  $\varepsilon(x) \equiv |Wx|$ . Clearly we have

$$\varepsilon(x) = \frac{|W|}{h_x}. \tag{8}$$

**Proposition 2.2.**

(i) *For any  $x \in \mathbb{R}^n/Q^\vee$  there exist  $x' \in F \cap \mathbb{R}^n/Q^\vee$  and  $w \in W$  such that*

$$x = wx'. \tag{9}$$

(ii) *If  $x, x' \in F \cap \mathbb{R}^n/Q^\vee$  and  $x' = wx, w \in W$ , then*

$$x' = x = wx. \tag{10}$$

(iii) *If  $x \in F \cap \mathbb{R}^n/Q^\vee$ , i.e.  $x = a + Q^\vee, a \in F$ , then*

$$\text{Stab}(x) \cong \text{Stab}_{W_{\text{aff}}}(a). \tag{11}$$

**Table 1.** The marks and the dual marks of simple Lie algebras. Numbering of simple roots as shown in figure 1 is used.

Type	Marks $m_1, \dots, m_n$	Dual marks $m_1^\vee, \dots, m_n^\vee$
$A_n (n \geq 1)$	1, 1, ..., 1	1, 1, ..., 1
$B_n (n \geq 3)$	1, 2, 2, ..., 2	2, 2, ..., 2, 1
$C_n (n \geq 2)$	2, 2, ..., 2, 1	1, 2, 2, ..., 2
$D_n (n \geq 4)$	1, 2, ..., 2, 1, 1	1, 2, ..., 2, 1, 1
$E_6$	1, 2, 3, 2, 1, 2	1, 2, 3, 2, 1, 2
$E_7$	2, 3, 4, 3, 2, 1, 2	2, 3, 4, 3, 2, 1, 2
$E_8$	2, 3, 4, 5, 6, 4, 2, 3	2, 3, 4, 5, 6, 4, 2, 3
$G_2$	2, 3	3, 2
$F_4$	2, 3, 4, 2	2, 4, 3, 2

**Proof.**

- (i) Follows directly from (5).
- (ii) Follows directly from (6).
- (iii) We have from (3) that for any  $w^{\text{aff}} \in W^{\text{aff}}$  there exist unique  $w \in W$  and unique shift  $T(q^\vee)$  such that  $w^{\text{aff}} = T(q^\vee)w$ . Define the map  $\psi : \text{Stab}_{W^{\text{aff}}}(a) \rightarrow W$  for  $w^{\text{aff}} \in \text{Stab}_{W^{\text{aff}}}(a)$  by

$$\psi(w^{\text{aff}}) = \psi(T(q^\vee)w) = w.$$

Since the relation

$$\psi(T(q_1^\vee)w_1 T(q_2^\vee)w_2) = \psi(T(q_1^\vee + w_1 q_2^\vee)w_1 w_2) = w_1 w_2$$

holds,  $\psi$  is indeed a homomorphism. If  $a = w^{\text{aff}}a = wa + q^\vee$ , then  $a - wa = q^\vee \in Q^\vee$ , i.e. we obtain  $w \in \text{Stab}(x)$  and vice versa. Thus,  $\psi(\text{Stab}_{W^{\text{aff}}}(a)) = \text{Stab}(x)$  holds. We also have

$$\ker \psi = \{T(q^\vee) \in \text{Stab}_{W^{\text{aff}}}(a)\} = 1.$$

□

**2.5. Dual Lie algebra**

The set of simple dual roots  $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$  is a system of simple roots of some simple Lie algebra. The system  $\Delta^\vee$  also spans the Euclidean space  $\mathbb{R}^n$ .

The dual system  $\Delta^\vee$  determines the following.

- The highest dual root  $\eta \equiv -\alpha_0^\vee = m_1^\vee \alpha_1^\vee + \dots + m_n^\vee \alpha_n^\vee$ . Here the coefficients  $m_j^\vee$  are called the dual marks of  $G$ . The marks and the dual marks are summarized in table 1.
- The dual Cartan matrix

$$C_{ij}^\vee = \frac{2\langle \alpha_i^\vee, \alpha_j^\vee \rangle}{\langle \alpha_j^\vee, \alpha_j^\vee \rangle} = C_{ji}, \quad i, j \in \{1, \dots, n\}.$$

- The dual root lattice

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee.$$

- The root lattice

$$Q = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n, \quad \text{where } \alpha_i = \frac{2\alpha_i^\vee}{\langle \alpha_i^\vee, \alpha_i^\vee \rangle}.$$



- The  $\mathbb{Z}$ -dual lattice

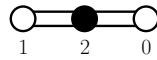
$$P = \{\omega \in \mathbb{R}^n \mid \langle \omega, \alpha^\vee \rangle \in \mathbb{Z}, \forall \alpha^\vee \in \Delta^\vee\} = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_n.$$

- The extended dual Coxeter–Dynkin diagram (DD $^\vee$ ) of  $G$  describes the system of dual roots  $\Delta^\vee$  together with the highest dual root  $\eta = -\alpha_0^\vee$ . The  $k$ th node corresponds to the vector  $\alpha_k^\vee, k = 0, \dots, n$ . The rules related to relative angles and lengths of vectors are the same as for extended diagrams. Note that the squared lengths of the white nodes (long dual roots)  $\alpha^\vee$  are determined by the formula

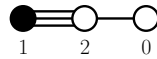
$$\langle \alpha^\vee, \alpha^\vee \rangle = \frac{4}{\langle \alpha, \alpha \rangle},$$

and in general  $\langle \alpha^\vee, \alpha^\vee \rangle \neq 2$ . For the extended DD $^\vee$  the following cases may occur.

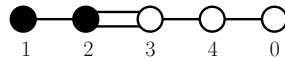
- (i) For the algebras  $A_n, D_n, E_6, E_7$  and  $E_8$  extended DD $^\vee$  coincide with the corresponding extended DD shown in figure 1.
- (ii) Extended DD $^\vee$  of  $B_n, n \geq 3$ , is the extended DD of  $C_n$  and vice versa.
- (iii) Extended DD $^\vee$  of  $C_2$  with unchanged numbering of the nodes.



- (iv) Extended DD $^\vee$  of  $G_2$  with unchanged numbering of the nodes.



- (v) Extended DD $^\vee$  of  $F_4$  with unchanged numbering of the nodes.



### 2.6. Dual affine Weyl group

The group generated by reflections with respect to dual roots  $r_{\alpha^\vee} = r_\alpha, \alpha^\vee \in \Delta^\vee$ , coincides with the Weyl group  $W$ . The dual affine Weyl group  $\widehat{W}^{\text{aff}}$  is generated by shifts from  $Q$  and the Weyl group  $W$ . Moreover,  $\widehat{W}^{\text{aff}}$  is a semidirect product of group of shifts  $Q$  and the Weyl group  $W$  :

$$\widehat{W}^{\text{aff}} = Q \rtimes W. \tag{12}$$

Equivalently,  $\widehat{W}^{\text{aff}}$  is generated by the reflections  $r_i$  and reflection  $r_0^\vee$ , where

$$r_0^\vee a = r_\eta a + \frac{2\eta}{\langle \eta, \eta \rangle}, \quad r_\eta a = a - \frac{2\langle a, \eta \rangle}{\langle \eta, \eta \rangle} \eta, \quad a \in \mathbb{R}^n.$$

The fundamental region  $F^\vee$  of  $\widehat{W}^{\text{aff}}$  is the convex hull of the vertices  $\{0, \frac{\omega_1}{m_1^\vee}, \dots, \frac{\omega_n}{m_n^\vee}\}$ :

$$F^\vee = \{z_1\omega_1 + \dots + z_n\omega_n \mid z_0, \dots, z_n \in \mathbb{R}^{\geq 0}, z_0 + z_1m_1^\vee + \dots + z_nm_n^\vee = 1\} \\ = \{a \in \mathbb{R}^n \mid \langle a, \alpha^\vee \rangle \geq 0, \forall \alpha^\vee \in \Delta^\vee, \langle a, \eta \rangle \leq 1\} \tag{13}$$

and

$$\widehat{W}^{\text{aff}} F^\vee = \mathbb{R}^n.$$

Since  $F^\vee$  is a fundamental region, we have the following.

- (i) For any  $a \in \mathbb{R}^n$  there exist  $a' \in F^\vee, w \in W$  and  $q \in Q$  such that

$$a = wa' + q. \tag{14}$$

- (ii) If  $a, a' \in F^\vee$  and  $a' = w^{\text{aff}} a, w^{\text{aff}} \in \widehat{W}^{\text{aff}}$ , then  $a = a'$ , i.e. if there exist  $w \in W$  and  $q \in Q$  such that  $a' = wa + q$ , then

$$a' = a = wa + q. \tag{15}$$

- (iii) Consider the point  $a = z_1\omega_1 + \dots + z_n\omega_n \in F^\vee$  such that  $z_0 + z_1m_1^\vee + \dots + z_nm_n^\vee = 1$ . The isotropy group

$$\text{Stab}_{\widehat{W}^{\text{aff}}}(a) = \{w^{\text{aff}} \in \widehat{W}^{\text{aff}} \mid w^{\text{aff}} a = a\} \tag{16}$$

of the point  $a$  is trivial,  $\text{Stab}_{\widehat{W}^{\text{aff}}}(a) = 1$ , if  $a \in \text{int}(F^\vee)$ , i.e. all  $z_i > 0, i = 0, \dots, n$ . Otherwise the group  $\text{Stab}_{\widehat{W}^{\text{aff}}}(a)$  is generated by such  $r_i^\vee$  for which  $z_i = 0, i = 0, \dots, n$ . The Coxeter–Dynkin diagram of  $\text{Stab}_{\widehat{W}^{\text{aff}}}(a)$  is obtained as a subgraph of extended  $\text{DD}^\vee$  by omitting such vertices  $\alpha_i^\vee$  (and adjacent edges) for which  $z_i > 0, i = 0, \dots, n$ . Since there always exists some  $z_i > 0, i = 0, \dots, n$ , this subgraph is always proper.

### 3. Grids $F_M$ and $\Lambda_M$

#### 3.1. Grid $F_M$

The grid  $F_M$  is the finite fragment of the lattice  $\frac{1}{M}P^\vee$  which is found inside of  $F$ . Suppose we have fixed  $M \in \mathbb{N}$  and consider the  $W$ -invariant group  $\frac{1}{M}P^\vee/Q^\vee$  of the finite order:

$$\left| \frac{1}{M}P^\vee/Q^\vee \right| = cM^n. \tag{17}$$

We define the grid  $F_M$  as such cosets from  $\frac{1}{M}P^\vee/Q^\vee$  which have representative elements in the fundamental domain  $F$ :

$$F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F.$$

From relation (9) we have that

$$WF_M = \frac{1}{M}P^\vee/Q^\vee. \tag{18}$$

We have immediately from (4) that the set  $F_M$ , or more precisely its representative points, can be identified as

$$F_M = \left\{ \frac{s_1}{M}\omega_1^\vee + \dots + \frac{s_n}{M}\omega_n^\vee \mid s_0, s_1, \dots, s_n \in \mathbb{Z}^{\geq 0}, s_0 + s_1m_1 + \dots + s_nm_n = M \right\}. \tag{19}$$

Another interpretation of the points of  $F_M$  is as points representing the conjugacy classes of a finite Abelian subgroup of the maximal torus which is generated by the elements of order  $M$  of the torus. The order of that group can be determined after the number of elements of  $F_M$  is found in the next subsection.

#### 3.2. Number of elements of $F_M$

The number of elements of  $F_M$  could be obtained by putting together the results of Djoković [18, 19], namely the numbers corresponding of all divisors of  $M$ . We proceed in another way by determining the number directly.

**Proposition 3.1.** *Let  $K, l \in \mathbb{Z}^{\geq 0}$ . Then the number of solutions of the equation*

$$a_0 + a_1 + \dots + a_l = K, \quad a_0, \dots, a_l \in \mathbb{Z}^{\geq 0}, \tag{20}$$

*is equal to  $\binom{l+K}{l}$ .*

**Proof.** We proceed by induction on  $l$ . If  $l = 0$ , then the equation  $a_0 = K$  has exactly one solution. We arrange the solutions of (20) into  $K + 1$  disjoint subsets according the value of

$a_l = 0, \dots, K$ . According to the assumption, the number of solutions in each of these subsets is equal to  $\binom{l-1+K-a_l}{l-1}$ . Then we have

$$\sum_{a_l=0}^K \binom{l-1+K-a_l}{l-1} = \sum_{k=0}^K \binom{l-1+k}{l-1} = \binom{l+K}{l}. \quad \square$$

**Proposition 3.2.** Let  $n, m_1, \dots, m_n, M \in \mathbb{N}$ , denote  $L = \text{lcm}(m_1, \dots, m_n)$  and suppose

$$M = Lk + l, \quad k, l \in \mathbb{Z}^{\geq 0}, \quad l < L, \\ (n+1)L - (1+m_1+\dots+m_n) = LN + N', \quad N, N' \in \mathbb{Z}^{\geq 0}, \quad N' < L.$$

Then the number of solutions of the equation

$$a_0 + m_1 a_1 + \dots + m_n a_n = M, \quad a_0, \dots, a_n \in \mathbb{Z}^{\geq 0}, \quad (21)$$

is equal to  $\sum_{i=0}^N d_{li} \binom{n-i+k}{n}$ , where  $d_{li}$  is the number of solutions of the equation

$$l_0 + m_1 l_1 + \dots + m_n l_n = Li + l, \quad l_0 \in \{0, \dots, L-1\}, \\ l_1 \in \left\{0, \dots, \frac{L}{m_1} - 1\right\}, \dots, l_n \in \left\{0, \dots, \frac{L}{m_n} - 1\right\}. \quad (22)$$

**Proof.** Suppose we have fixed  $M = Lk + l, k, l \in \mathbb{Z}^{\geq 0}, l < L$ , and denote by  $[a_0, a_1, \dots, a_n]$  a solution of (21). Consider the mapping

$$[a_0, a_1, \dots, a_n] \mapsto [[l_0, l_1, \dots, l_n], [k_0, k_1, \dots, k_n]]$$

via the relations

$$a_0 = Lk_0 + l_0, \quad k_0, l_0 \in \mathbb{Z}^{\geq 0}, \quad l_0 < L \\ a_1 = \frac{L}{m_1} k_1 + l_1, \quad k_1, l_1 \in \mathbb{Z}^{\geq 0}, \quad l_1 < \frac{L}{m_1} \\ \vdots \\ a_n = \frac{L}{m_n} k_n + l_n, \quad k_n, l_n \in \mathbb{Z}^{\geq 0}, \quad l_n < \frac{L}{m_n}. \quad (23)$$

Substituting (23) into (21) we obtain that  $[l_0, l_1, \dots, l_n]$  is a solution of (22) for some  $i \in \{0, 1, \dots, N\}$  such that

$$k_0 + k_1 + \dots + k_n = k - i. \quad (24)$$

Conversely, for fixed  $i \in \{0, 1, \dots, N\}$ , each solution  $[k_0, k_1, \dots, k_n]$  of (24) together with a solution of (22) leads to some solution of (21), i.e. we have a decomposition of the solutions of (21) into disjoint subsets  $S_i, i \in \{0, 1, \dots, N\}$ . Taking into account that the number of solutions of (24) is  $\binom{n+k-i}{n}$ , we have  $|S_i| = d_{li} \binom{n+k-i}{n}$ .  $\square$

**Theorem 3.3.** The numbers of points of the grid  $F_M$  of Lie algebras  $A_n, B_n, C_n, D_n$  are given by the following relations.

(i)  $A_n, n \geq 1$ ,

$$|F_M(A_n)| = \binom{n+M}{n}$$

(ii)  $C_n, n \geq 2$ ,

$$|F_{2k}(C_n)| = \binom{n+k}{n} + \binom{n+k-1}{n}$$

$$|F_{2k+1}(C_n)| = 2 \binom{n+k}{n}$$

(iii)  $B_n, n \geq 3,$

$$|F_M(B_n)| = |F_M(C_n)|$$

(iv)  $D_n, n \geq 4,$

$$|F_{2k}(D_n)| = \binom{n+k}{n} + 6 \binom{n+k-1}{n} + \binom{n+k-2}{n}$$

$$|F_{2k+1}(D_n)| = 4 \binom{n+k}{n} + 4 \binom{n+k-1}{n}$$

**Proof.**

(i) Follows directly from proposition 3.1.

(ii) We have  $m_1 = 2, m_2 = 2, \dots, m_n = 1$  and thus,  $L = 2$  and Coxeter number  $m = 2n, N = 1.$

(i)  $M = 2k.$  According to proposition 3.2 we have to analyze the number of solutions of the equation

$$l_0 + l_n = 2i, \quad l_0 \in \{0, 1\}, \quad l_n \in \{0, 1\},$$

for  $i \in \{0, 1\}.$  This equation has one solution  $[0, 0]$  for  $i = 0$  and one solution  $[1, 1]$  for  $i = 1.$

(ii)  $M = 2k + 1.$  The equation

$$l_0 + l_n = 2i + 1, \quad l_0 \in \{0, 1\}, \quad l_n \in \{0, 1\},$$

has two solutions  $[0, 1], [1, 0]$  for  $i = 0$  and no solution for  $i = 1.$

The proof of the cases  $B_n, D_n$  is analogous. □

There is a convenient way of writing the expressions for  $|F_M|.$  For that we introduce, for a given Lie algebra, a matrix

$$R = (d_{li}), \quad l \in \{0, \dots, L - 1\}, \quad i \in \{0, \dots, N\},$$

where the numbers  $d_{li}$  and the integers  $L, N$  were defined in proposition 3.2. The values of  $L, N$  are summarized in table 2. The matrix  $R$  then determines the number of points of  $F_M$  via the relation

$$|F_{Lk+l}| = \sum_{i=0}^N R_{li} \binom{n-i+k}{n}. \tag{25}$$

The matrices  $R$  for all simple Lie algebras are listed in the appendix.

**Example 3.1.** We demonstrate the application of formula (25) for the case  $E_8.$  We take the sequence of numbers in the second row of the matrix  $R(E_8)$  in the appendix, i.e. the numbers 1, 1520913, 151753498, ..., and calculate

$$|F_{60k+1}(E_8)| = \binom{8+k}{8} + 1520913 \binom{7+k}{8} + 151753498 \binom{6+k}{8} + 1668997388 \binom{5+k}{8}$$

$$+ 4262344515 \binom{4+k}{8} + 3044716089 \binom{3+k}{8} + 571669092 \binom{2+k}{8}$$

$$+ 18981162 \binom{1+k}{8} + 17342 \binom{k}{8}$$

$$= \frac{1}{84} (2k+1)(3k+1)(5k+1)(5k+2)(10k+3)(15k+2)(30k+1)(30k+7)$$

**Table 2.** Orders of the finite Weyl groups  $|W|$ , volumes of  $F$ , orders of the center of the compact Lie group, Coxeter numbers  $m$ , numbers  $L, N$ .

	$A_n(n \geq 1)$	$B_n(n \geq 3)$	$C_n(n \geq 2)$	$D_n(n \geq 4)$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$ W $	$(n+1)!$	$2^n n!$	$2^n n!$	$2^{n-1} n!$	$2^7 3^4 5$	$2^{10} 3^4 5 \cdot 7$	$2^{14} 3^5 5^2 7$	$2^7 3^2$	12
$\text{vol}(F)$	$\frac{1}{n! \sqrt{n+1}}$	$\frac{1}{n! 2^{n-1}}$	$\frac{1}{n! 2^{\frac{n}{2}}}$	$\frac{1}{n! 2^{n-2}}$	$\frac{\sqrt{3}}{2^7 3^4 5}$	$\frac{\sqrt{2}}{2^{10} 3^4 5 \cdot 7}$	$\frac{1}{2^{14} 3^5 5^2 7}$	$\frac{1}{2^6 3^2}$	$\frac{\sqrt{3}}{12}$
$c$	$n+1$	2	2	4	3	2	1	1	1
$m$	$n+1$	$2n$	$2n$	$2n-2$	12	18	30	12	6
$L$	1	2	2	2	6	12	60	12	6
$N$	0	1	1	2	5	6	8	4	2

Thus, we have

$$\begin{aligned}
 |F_1(E_8)| &= 1 \\
 |F_{61}(E_8)| &= 1520\,922 \\
 |F_{121}(E_8)| &= 165\,441\,760 \\
 |F_{181}(E_8)| &= 3103\,220120 \\
 &\vdots
 \end{aligned}$$

The polynomial  $|F_{60k+1}(E_8)|$  has real roots, and it could be written in factorized form. The following polynomial can be calculated similarly, but its roots are not real; therefore, we do not factorize it:

$$\begin{aligned}
 |F_{60k+30}(E_8)| &= \frac{1}{56} (13\,500\,000k^8 + 81\,000\,000k^7 + 211\,575\,000k^6 + 314\,212\,500k^5 + \\
 &\quad + 290\,172\,925k^4 + 170\,628\,150k^3 + 62\,388\,155k^2 + 12\,968\,910k + 1173\,536)
 \end{aligned}$$

and we obtain a sequence

$$\begin{aligned}
 |F_{30}(E_8)| &= 20\,956 \\
 |F_{90}(E_8)| &= 20\,671\,771 \\
 |F_{150}(E_8)| &= 780\,429\,571 \\
 |F_{210}(E_8)| &= 9\,375\,443\,806 \\
 &\vdots
 \end{aligned}$$

### 3.3. Grid $F_M$ and elements of finite order in $G$

The grid  $F_M$  consists of points representing the elements of Ad-order  $M$  and of Ad-order equal to its divisors [15, 16, 22]. The elements of Ad-order  $M$  only are found by additional requirement  $\text{gcd}(s_0, s_1, \dots, s_n) = 1$  in (19). Each point then represents particular conjugacy class of elements of Ad-order  $M$ . The relation between (full) order  $k$  of elements in a conjugacy class of Ad-order  $M$  is given  $k = Mj$ , where explicit formulas for  $j$  are listed in table 6 in [15].

In [18, 19] were calculated explicit formulas for the number  $\nu(k, G)$  of conjugacy classes of elements whose order divide  $k$ , in any complex semisimple Lie group  $G$ . We have for example in the case of  $G = C_n, n \geq 2$ ,

$$\nu(2k, C_n) = \nu(2k+1, C_n) = \binom{n+k}{n}.$$

**Table 3.** Sets  $F_M(C_2)$  for  $M = 1, 2, 3, 6$ . Each point of  $F_M(C_2)$  is represented by the coordinates  $[s_0, s_1, s_2]$  from (19). The third column contains the full order  $k$  of the corresponding conjugacy class of the elements of finite order.

	$[s_0, s_1, s_2]$	$\gcd(s_0, s_1, s_2)$	$k$
$M = 1$	[1, 0, 0]	1	1
	[0, 0, 1]	1	2
$M = 2$	[0, 1, 0]	1	2
	[1, 0, 1]	1	4
	[2, 0, 0]	2	–
	[0, 0, 2]	2	–
$M = 3$	[1, 1, 0]	1	3
	[1, 0, 2]	1	3
	[2, 0, 1]	1	6
	[0, 1, 1]	1	6
	[3, 0, 0]	3	–
	[0, 0, 3]	3	–
$M = 6$	[4, 1, 0]	1	6
	[2, 1, 2]	1	6
	[0, 1, 4]	1	6
	[5, 0, 1]	1	12
	[3, 1, 1]	1	12
	[1, 2, 1]	1	12
	[1, 1, 3]	1	12
	[1, 0, 5]	1	12
	[2, 2, 0]	2	–
	[2, 0, 4]	2	–
	[4, 0, 2]	2	–
	[0, 2, 2]	2	–
	[0, 3, 0]	3	–
	[3, 0, 3]	3	–
	[6, 0, 0]	6	–
	[0, 0, 6]	6	–

**Example 3.2.** Consider the case  $C_2$ . We have from [15] that  $j = 2/\gcd(2, s_2)$ . In table 3 are listed 2, 4, 6 and 16 elements of  $F_1(C_2)$ ,  $F_2(C_2)$ ,  $F_3(C_2)$  and  $F_6(C_2)$ , respectively. Among these 28 elements there are  $\nu(6, C_2) = \binom{2+3}{2} = 10$  elements whose order  $k$  divides 6.

Comparing the explicit formulas for  $\nu(k, G)$  from [19] with our formulas for  $|F_M(G)|$ , one finds a similarity which suggests some relation between them. In the following proposition, we single out such cases of  $G$  for which these formulas coincide. Note also that the following proposition holds for precisely the simple Lie groups with the center of order 1.

**Proposition 3.4.** *If  $G = G_2, F_4$  or  $E_8$ , then for all  $M \in \mathbb{N}$*

$$|F_M(G)| = \nu(M, G).$$

**Proof.** For  $K|M$  define the set  $F_{M,K} \subset F_M$  by

$$F_{M,K} = \left\{ \frac{s_1}{M} \omega_1^\vee + \dots + \frac{s_n}{M} \omega_n^\vee \mid s_0, s_1, \dots, s_n \in \mathbb{Z}^{\geq 0}, s_0 + \sum_{i=1}^n m_i s_i = M, \gcd(s_0, s_1, \dots, s_n) = K \right\}.$$

Since one can write the set  $F_M$  as a disjoint union of the sets  $F_{M,K}$  with  $K|M$ , we have

$$|F_M| = \sum_{K|M} |F_{M,K}|. \tag{26}$$

There is one-to-one correspondence between the set  $F_{M,K}$  and the set  $F_{\frac{M}{K},1}$  via the mapping

$$F_{\frac{M}{K},1} \ni y \mapsto Ky \in F_{M,K}$$

and thus we have

$$|F_{M,K}| = |F_{\frac{M}{K},1}|. \tag{27}$$

Since for  $G = G_2, F_4$  and  $E_8$ , we have from [15] that  $j = 1$ , and the number of conjugacy classes of order  $k$  is equal to  $|F_{k,1}|$ . Thus, using (26), (27) we obtain

$$|F_M(G)| = \sum_{K|M} |F_{M,K}| = \sum_{K|M} |F_{\frac{M}{K},1}| = \sum_{K|M} |F_{K,1}| = v(M, G). \quad \square$$

### 3.4. Interior of $F_M$

Subsequently, we will need to know the order of the stabilizer of a given point in the grid  $F_M$ . The stabilizer is trivial if the element is found in the interior of  $F_M$ . Below we consider special functions called  $S$ -functions and their discrete pairwise orthogonality. Only the points from the interior of the  $F_M$  are involved in the orthogonality relations of  $S$ -functions. Define an interior of the grid  $F_M$  as such cosets from  $\frac{1}{M}P^\vee/Q^\vee$  which have representative element in the interior of  $F$ :

$$\tilde{F}_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap \text{int}(F).$$

Points from  $F_M$  which have all coordinates  $s_0, s_1, \dots, s_n$  positive are precisely those from  $\tilde{F}_M$ . Thus, from (19) we see that the set  $\tilde{F}_M$  can be identified as

$$\tilde{F}_M = \left\{ \frac{s'_1}{M} \omega_1^\vee + \dots + \frac{s'_n}{M} \omega_n^\vee \mid s'_0, s'_1, \dots, s'_n \in \mathbb{N}, s'_0 + s'_1 m_1 + \dots + s'_n m_n = M \right\}. \tag{28}$$

We calculate the number of elements of  $\tilde{F}_M$ .

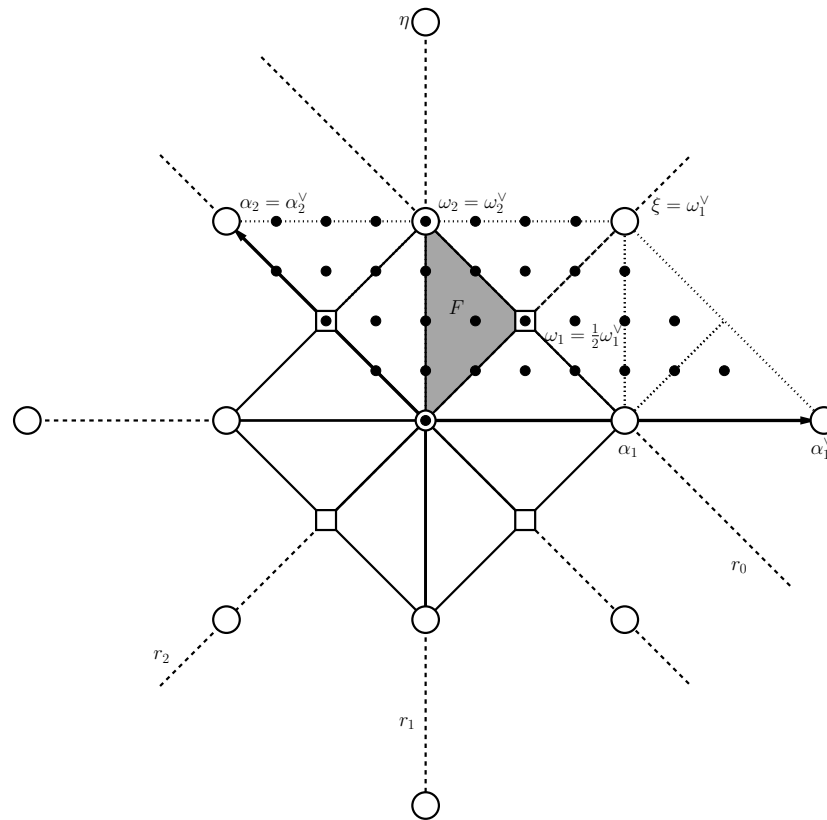
**Proposition 3.5.** *Let  $m$  be the Coxeter number. Then*

$$|\tilde{F}_M| = \begin{cases} 0 & M < m \\ 1 & M = m \\ |F_{M-m}| & M > m. \end{cases}$$

**Proof.** Substituting the relations  $s'_i = 1 + s_i, s_i \in \mathbb{Z}^{\geq 0}, i \in \{0, \dots, n\}$ , into the defining relation (28), we obtain

$$s_0 + m_1 s_1 + \dots + m_n s_n = M - m, \quad s_0, \dots, s_n \in \mathbb{Z}^{\geq 0}.$$

This equation has one solution  $[0, \dots, 0]$  if  $M = m$ , no solution if  $M < m$  and is equal to the defining relation (19) of  $F_{M-m}$  if  $M > m$ .  $\square$



**Figure 2.** The cosets representants of  $\frac{1}{4}P^\vee/Q^\vee$  of  $C_2$ ; the cosets representants are shown as 32 black dots, and the grey area is the fundamental domain  $F$  which contains nine points of  $F_4(C_2)$ . The dashed lines represent ‘mirrors’  $r_0, r_1$  and  $r_2$ . Circles are elements of the root lattice  $Q$ , and together with squares they are elements of the weight lattice  $P$ .

**Example 3.3.** For the algebra  $E_8$  we have the Coxeter number  $m = 30$ . We calculated in example 3.1 that  $|F_{30}(E_8)| = 20956$ . According to proposition 3.5, among these 20956 points, there is only one in the interior of  $F_{30}(E_8)$ , i.e.  $|\tilde{F}_{30}(E_8)| = 1$ .

**Example 3.4.** For the Lie algebra  $C_2$  we have Coxeter numbers  $m = 4$  and  $c = 2$ . Consider for example  $M = 4$ . For the order of the group  $\frac{1}{4}P^\vee/Q^\vee$  we have from (17) that  $|\frac{1}{4}P^\vee/Q^\vee| = 32$  and according to theorem 3.3 we calculate

$$|F_4(C_2)| = \binom{4}{2} + \binom{3}{2} = 9.$$

Note also that from proposition 3.5 it follows that the number of points in the interior  $|\tilde{F}_4(C_2)| = 1$ . The cosets representants of  $\frac{1}{4}P^\vee/Q^\vee$  and fundamental domain  $F$  are depicted in figure 2.

### 3.5. Grid $\Lambda_M$

The points of  $\Lambda_M$  are the dominant weights specifying  $C$ - or  $S$ -functions which belong to the same pairwise orthogonal set. Later on we consider  $C$ - or  $S$ -functions which are sampled on



the points  $F_M$ . We consider the lowest possible set of such points. The number of points of  $\Lambda_M$  coincides with the number of points of  $F_M$ .

The  $W$ -invariant group  $P/MQ$  is isomorphic to the group  $\frac{1}{M}P^\vee/Q^\vee$ ; thus, its order is given by the formula

$$|P/MQ| = cM^n.$$

Define the grid  $\Lambda_M$  as such cosets from  $P/MQ$  which has some representative element in  $MF^\vee$ :

$$\Lambda_M \equiv MF^\vee \cap P/MQ.$$

We have from (13) that the set  $\Lambda_M$ , or more precisely its representative points, can be identified as

$$\Lambda_M = \{t_1\omega_1 + \dots + t_n\omega_n \mid t_0, t_1, \dots, t_n \in \mathbb{Z}^{\geq 0}, t_0 + t_1m_1^\vee + \dots + t_nm_n^\vee = M\}. \quad (29)$$

Observing table 1, we see that the  $n$ -tuple of dual marks  $(m_1^\vee, \dots, m_n^\vee)$  is a certain permutation of the  $n$ -tuple  $(m_1, \dots, m_n)$ . Thus, we have from (19) and (29) that

$$|F_M| = |\Lambda_M|. \quad (30)$$

If we have two elements  $b, b' \in \mathbb{R}^n$  such that  $b' - b = Mq$ , with  $q \in Q$ , then for  $w \in W$  we have  $wb - wb' = wMq \in MQ$ , i.e. we have a natural action of  $W$  on the quotient group  $\mathbb{R}^n/MQ$ . For  $\lambda \in \mathbb{R}^n/MQ$  we denote

$$\text{Stab}^\vee(\lambda) = \{w \in W \mid w\lambda = \lambda\}$$

and

$$h_\lambda^\vee \equiv |\text{Stab}^\vee(\lambda)|. \quad (31)$$

**Proposition 3.6.**

(i) For any  $\lambda \in P/MQ$  there exist  $\lambda' \in \Lambda_M$  and  $w \in W$  such that

$$\lambda = w\lambda'. \quad (32)$$

(ii) If  $\lambda, \lambda' \in \Lambda_M$  and  $\lambda' = w\lambda$ ,  $w \in W$ , then

$$\lambda' = \lambda = w\lambda. \quad (33)$$

(iii) If  $\lambda \in MF^\vee \cap \mathbb{R}^n/MQ$ , i.e.  $\lambda = b + MQ$ ,  $b \in MF^\vee$ , then

$$\text{Stab}^\vee(\lambda) \cong \text{Stab}_{\widehat{W}^{\text{aff}}}(b/M). \quad (34)$$

**Proof.**

(i) Let  $\lambda \in P/MQ$  be of the form  $\lambda = p + MQ$ ,  $p \in P$ . From (14) it follows that there exist  $p' \in F^\vee$ ,  $w \in W$  and  $q \in Q$  such that

$$\frac{1}{M}p = wp' + q,$$

i.e.  $p = wMp' + Mq$ . From  $W$ -invariance of  $P$  we have that  $Mp' \in P$ , the class  $\lambda' = Mp' + MQ$  is from  $\Lambda_M$  and (32) holds.

(ii) Let  $\lambda, \lambda' \in P/MQ$  be of the form  $\lambda = p + MQ$ ,  $\lambda' = p' + MQ$  and  $p, p' \in MF^\vee$ . Suppose that

$$p' = wp + Mq, \quad q \in Q, \quad w \in W.$$

Then  $p/M, p'/M \in F^\vee$  and it follows from (15) that  $p = p'$ .

(iii) We have from (12) that for any  $w^{\text{aff}} \in \widehat{W}^{\text{aff}}$  there exist unique  $w \in W$  and unique shift  $T(q)$  such that  $w^{\text{aff}} = T(q)w$ . Define map  $\psi : \text{Stab}_{\widehat{W}^{\text{aff}}}(b/M) \rightarrow W$  for  $w^{\text{aff}} \in \text{Stab}_{\widehat{W}^{\text{aff}}}(b/M)$  by

$$\psi(w^{\text{aff}}) = \psi(T(q)w) = w.$$

Since the relation

$$\psi(T(q_1)w_1T(q_2)w_2) = \psi(T(q_1 + w_1q_2)w_1w_2) = w_1w_2$$

holds,  $\psi$  is indeed a homomorphism. If  $b/M = w^{\text{aff}}(b/M) = w(b/M) + q$ , then  $b - wb = Mq \in MQ$ , i.e. we obtain  $w \in \text{Stab}^\vee(\lambda)$  and vice versa. Thus,  $\psi(\text{Stab}_{\widehat{W}^{\text{aff}}}(b/M)) = \text{Stab}^\vee(\lambda)$  holds. We also have

$$\ker \psi = \{T(q) \in \text{Stab}_{\widehat{W}^{\text{aff}}}(b/M)\} = 1. \quad \square$$

### 3.6. Interior of $\Lambda_M$

The points of the interior of  $\Lambda_M$  are the points which label non-zero  $S$ -functions. For the points on the boundary of  $\Lambda_M$ , the  $S$ -functions are equal to zero. Define an interior of the grid  $\Lambda_M$  as such cosets from  $P/MQ$  which have representative element in the interior of  $MF^\vee$ :

$$\widetilde{\Lambda}_M \equiv P/MQ \cap \text{int}(MF^\vee).$$

Analogously to (28), the set  $\widetilde{\Lambda}_M$  can be identified as

$$\widetilde{\Lambda}_M = \{t'_1\omega_1 + \dots + t'_n\omega_n \mid t'_0, t'_1, \dots, t'_n \in \mathbb{N}, t'_0 + t'_1m_1^\vee + \dots + t'_nm_n^\vee = M\} \quad (35)$$

and moreover

$$|\widetilde{F}_M| = |\widetilde{\Lambda}_M|. \quad (36)$$

**Example 3.5.** For the Lie algebra  $C_2$  we have  $|P/4Q| = 32$  and according to (30) we have

$$|\Lambda_4(C_2)| = |F_4(C_2)| = 9.$$

Note also that from (36) it follows that the number of points in the interior  $|\widetilde{\Lambda}_4(C_2)| = 1$ . The cosets representants of  $P/4Q$ , the dual fundamental domain  $F^\vee$  and the grid  $\Lambda_4(C_2) = 4F^\vee \cap P/4Q$  are depicted in figure 3.

### 3.7. Calculation of $h_x$ and $h_\lambda^\vee$

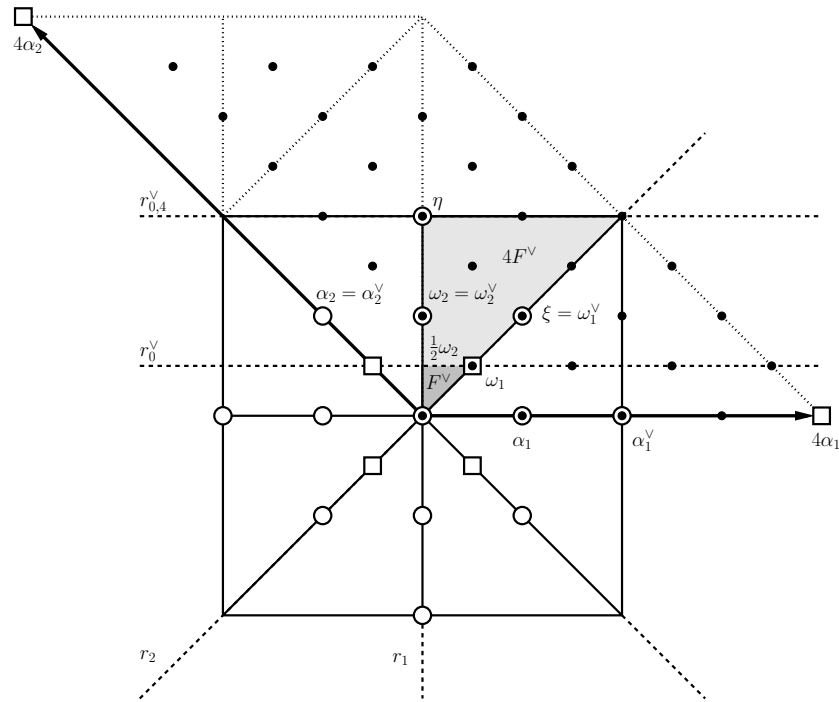
Calculation procedure of  $h_x$  for any  $x \in F_M$  can be deduced from (7) and (11). It uses the extended DD of  $G$ . Using the extended  $\text{DD}^\vee$  of  $G$ , we obtain from (16) and (34) analogous calculation procedure for  $h_\lambda^\vee, \lambda \in \Lambda_M$ .

Consider a point  $x \in F_M$  and its corresponding coordinates  $[s_0, \dots, s_n]$  from (19).

- (i) If  $s_0, \dots, s_n$  are all non-zero, then  $h_x = 1$ .
- (ii) Consider such a subgraph  $U$  of extended DD consisting only of those nodes  $i$  for which  $s_i = 0, i = 0, \dots, n$ . The subgraph  $U$  consists in general of several connected components  $U_l$ . Each component  $U_l$  is a (non-extended) DD of some simple Lie group  $G_l$ . Take corresponding orders of the Weyl groups  $|W_l|$  of  $G_l$  from table 2. Then  $h_x = \prod_l |W_l|$ .

We proceed similarly to determine  $h_\lambda^\vee$  when we consider a point  $\lambda \in \Lambda_M$  and its corresponding coordinates  $[t_0, \dots, t_n]$  from (29).

- (i) If  $t_0, \dots, t_n$  are all non-zero, then  $h_\lambda^\vee = 1$ .
- (ii) Consider such a subgraph  $U'$  of the extended  $\text{DD}^\vee$  consisting only of those nodes  $i$  for which  $t_i = 0, i = 0, \dots, n$ . The subgraph  $U'$  consists in general of several connected components  $U'_l$ . Each component  $U'_l$  is a (non-extended) DD of some simple Lie group  $G'_l$ . Take corresponding orders of the Weyl groups  $|W'_l|$  of  $G'_l$  from table 2. Then  $h_\lambda^\vee = \prod_l |W'_l|$ .



**Figure 3.** The cosets representants of  $P/4Q$  of  $C_2$ ; the cosets representants are shown as 32 black dots, the darker grey area is the fundamental domain  $F^V$  and the lighter grey area is the domain  $4F^V$  which contains nine elements of  $\Lambda_4(C_2)$ . The dashed lines represent dual ‘mirrors’  $r_0^V, r_1, r_2$  and the affine mirror  $r_{0,4}^V$  (see (47)). The circles and squares coincide with those in figure 2.

#### 4. $W$ -invariant functions

The numbers  $h_x, h_\lambda^V$  and  $|F_M|$ , which were determined so far, are important for the properties of special functions called  $C$ - and  $S$ -functions when they sampled on  $F_M$ . A detailed review of the properties of  $C$ - and  $S$ -functions may be found in [2, 3]. In this section we want to complete and make explicit the orthogonality properties of  $C$ - and  $S$ -functions [17]. Two- and three-dimensional examples of these relations are found in [5–9].

##### 4.1. $C$ -functions

We recall the definition of  $C$ -functions and show that they can be labeled by the finite set  $\Lambda_M$  when they are sampled on the grid  $F_M$ .

Consider  $b \in P$  and recall that (normalized)  $C$ -functions can be defined as a mapping  $\Phi_b : \mathbb{R}^n \rightarrow \mathbb{C}$

$$\Phi_b(a) = \sum_{w \in W} e^{2\pi i(wb, a)}. \tag{37}$$

The following properties of  $C$ -functions are crucial:

- symmetry with respect to  $w \in W$

$$\Phi_b(wa) = \Phi_b(a) \tag{38}$$

$$\Phi_{wb}(a) = \Phi_b(a), \tag{39}$$

- invariance with respect to  $q^\vee \in Q^\vee$

$$\Phi_b(a + q^\vee) = \Phi_b(a). \tag{40}$$

We investigate the values of  $C$ -functions on the grid  $F_M$ . Suppose we have fixed  $M \in \mathbb{N}$  and  $s \in \frac{1}{M}P^\vee$ . From (40) it follows that we can consider  $\Phi_b$  as a function on classes  $\frac{1}{M}P^\vee/Q^\vee$ . From (9) and (38) it follows that we can consider  $\Phi_b$  only on the set  $F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F$ . We also have

$$\Phi_{b+MQ}(s) = \Phi_b(s), \quad s \in F_M,$$

and thus we can consider the functions  $\Phi_\lambda$  on  $F_M$  parameterized by classes from  $\lambda \in P/MQ$ . Moreover, from (32) and (39) it follows that we can consider  $C$ -functions  $\Phi_\lambda$  on  $F_M$  parameterized by  $\lambda \in \Lambda_M$  only.

#### 4.2. $S$ -functions

We recall the definition of  $S$ -functions and show that they can be labeled by the finite set  $\tilde{\Lambda}_M$  when they are sampled on the grid  $\tilde{F}_M$ .

Consider  $b \in P$  and recall that (normalized)  $S$ -functions can be defined as a mapping  $\varphi_b : \mathbb{R}^n \rightarrow \mathbb{C}$ :

$$\varphi_b(a) = \sum_{w \in W} (\det w) e^{2\pi i(wb, a)}. \tag{41}$$

The following properties of  $S$ -functions are crucial:

- antisymmetry with respect to  $w \in W$

$$\varphi_b(wa) = (\det w)\varphi_b(a) \tag{42}$$

$$\varphi_{wb}(a) = (\det w)\varphi_b(a), \tag{43}$$

- invariance with respect to  $q^\vee \in Q^\vee$

$$\varphi_b(a + q^\vee) = \varphi_b(a). \tag{44}$$

Moreover, simple calculation shows that  $\varphi_b(r_0a) = -\varphi_b(a)$  and thus we have

$$\varphi_b(r_i a) = -\varphi_b(a), \quad i \in \{0, \dots, n\}. \tag{45}$$

We investigate the behavior of  $S$ -functions on the grid  $F_M$ . Suppose we have fixed  $M \in \mathbb{N}$  and  $s \in \frac{1}{M}P^\vee$ . From (44) it follows that we can consider  $\varphi_b$  as a function on classes  $\frac{1}{M}P^\vee/Q^\vee$ . From (9) and (42) it follows that we can consider  $\varphi_b$  only on the set  $F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F$ . Moreover, from (45) we see that  $\varphi_b$  vanishes on the boundary of the fundamental domain  $F$  and thus we can consider  $\varphi_b$  only on the interior  $\tilde{F}_M$ .

We also have

$$\varphi_{b+MQ}(s) = \varphi_b(s), \quad s \in \tilde{F}_M,$$

and thus we can consider the functions  $\varphi_\lambda$  on  $\tilde{F}_M$  parameterized by classes from  $\lambda \in P/MQ$ . Moreover, from (32) and (43) it follows that we can consider  $\varphi_\lambda$  on  $\tilde{F}_M$  parameterized by classes from  $\Lambda_M$ . From (43) we have that

$$\varphi_{r_i^\vee \lambda}(s) = -\varphi_\lambda(s), \quad s \in \tilde{F}_M, \quad i \in \{1, \dots, n\}. \tag{46}$$

For the affine reflection  $r_{0,M}^\vee$ , given by

$$r_{0,M}^\vee \lambda = r_\eta \lambda + M \frac{2\eta}{\langle \eta, \eta \rangle} \tag{47}$$

we also obtain

$$\varphi_{r_{0,M}^\vee \lambda}(s) = -\varphi_\lambda(s), \quad s \in \tilde{F}_M. \tag{48}$$

We conclude that for  $\lambda \in \Lambda_M \setminus \tilde{\Lambda}_M$

$$\varphi_\lambda(s) = 0, \quad s \in \tilde{F}_M.$$

Thus, we can consider  $S$ -functions  $\varphi_\lambda$  on  $\tilde{F}_M$  parameterized by  $\lambda \in \tilde{\Lambda}_M$  only.

### 5. Discrete orthogonality of $C$ - and $S$ -functions

#### 5.1. Basic discrete orthogonality relations

Discrete orthogonality of the  $C$ - and  $S$ -functions was discussed in general in [17]. Practical use of [17] is not completely straightforward. Therefore we reformulate the basic facts and we subsequently use them to make the discrete orthogonality over  $F_M$  described in all details.

Note that since for  $a \in \frac{1}{M}P^\vee$  and  $b \in P$  the relation

$$e^{2\pi i(b+MQ, a+Q^\vee)} = e^{2\pi i(b, a)}$$

holds, the exponential mapping  $e^{2\pi i(\lambda, x)} \in \mathbb{C}$  for  $\lambda \in P/MQ$  and  $x \in \frac{1}{M}P^\vee/Q^\vee$  is well defined.

**Proposition 5.1.** *For all  $\lambda \in P/MQ, \lambda \neq 0$ , there exists  $x \in \frac{1}{M}P^\vee/Q^\vee$  such that  $e^{2\pi i(\lambda, x)} \neq 1$ .*

**Proof.** Suppose there is some  $p \in P$  and  $p \notin MQ$  such that for all  $p^\vee \in P^\vee$

$$\left\langle p, \frac{p^\vee}{M} \right\rangle = \left\langle \frac{p}{M}, p^\vee \right\rangle \in \mathbb{Z}. \tag{49}$$

Then from (2) we have  $p/M \in Q$ , i.e.  $p \in MQ$ —a contradiction. □

**Corollary 5.2.** *For any  $\lambda, \lambda' \in P/MQ$  it holds that*

$$\sum_{y \in \frac{1}{M}P^\vee/Q^\vee} e^{2\pi i(\lambda - \lambda', y)} = cM^n \delta_{\lambda, \lambda'}. \tag{50}$$

#### 5.2. Discrete orthogonality of $C$ -functions

We define the scalar product of two functions  $f, g : F_M \rightarrow \mathbb{C}$  by

$$\langle f, g \rangle_{F_M} = \sum_{x \in F_M} \varepsilon(x) f(x) \overline{g(x)}, \tag{51}$$

where the numbers  $\varepsilon(x)$  are determined by (8). We show that  $\Lambda_M$ , defined by (29), is the lowest maximal set of pairwise orthogonal  $C$ -functions.

**Proposition 5.3.** *For  $\lambda, \lambda' \in \Lambda_M$  it holds that*

$$\langle \Phi_\lambda, \Phi_{\lambda'} \rangle_{F_M} = c|W|M^n h_\lambda^\vee \delta_{\lambda, \lambda'}, \tag{52}$$

where  $c, h_\lambda^\vee, \Phi_\lambda$  were defined by (1), (31), (37) respectively, and  $n$  is the rank of  $G$ .

**Proof.** The equality

$$\sum_{x \in F_M} \varepsilon(x) \Phi_\lambda(x) \overline{\Phi_{\lambda'}(x)} = \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} \Phi_\lambda(y) \overline{\Phi_{\lambda'}(y)}$$

follows from (10) and (18) and  $W$ -invariance of the expression  $\Phi_\lambda(x) \overline{\Phi_{\lambda'}(x)}$ . Then, using  $W$ -invariance of  $\frac{1}{M} P^\vee / Q^\vee$ , we have

$$\begin{aligned} \langle \Phi_\lambda, \Phi_{\lambda'} \rangle_{F_M} &= \sum_{w' \in W} \sum_{w \in W} \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} e^{2\pi i(w\lambda - w'\lambda', y)} = |W| \sum_{w' \in W} \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} e^{2\pi i(\lambda - w'\lambda', y)} \\ &= c|W|M^n \sum_{w' \in W} \delta_{w'\lambda', \lambda}. \end{aligned}$$

Since  $\lambda, \lambda' \in \Lambda_M$ , we have from (33) that

$$\sum_{w' \in W} \delta_{w'\lambda', \lambda} = h_\lambda^\vee \delta_{\lambda, \lambda'}. \quad \square$$

### 5.3. Discrete orthogonality of $S$ -functions

Unlike the  $C$ -functions, the  $S$ -functions have non-zero values on the interior  $\tilde{F}_M$  only. Note that from (7) and (11) we have that  $\varepsilon(x) = |W|, x \in \tilde{F}_M$ . Therefore, we define the scalar product of two functions  $f, g : \tilde{F}_M \rightarrow \mathbb{C}$  as follows:

$$\langle f, g \rangle_{\tilde{F}_M} = |W| \sum_{x \in \tilde{F}_M} f(x) \overline{g(x)}. \quad (53)$$

Then since  $\varphi_\lambda$  vanishes on the boundary of  $F_M$ , we have

$$\langle \varphi_\lambda, \varphi_{\lambda'} \rangle_{F_M} = |W| \sum_{x \in \tilde{F}_M} \varphi_\lambda(x) \overline{\varphi_{\lambda'}(x)} = \langle \varphi_\lambda, \varphi_{\lambda'} \rangle_{\tilde{F}_M}.$$

We show that  $\tilde{\Lambda}_M$ , determined by (35), is the lowest maximal set of pairwise orthogonal  $S$ -functions.

**Proposition 5.4.** For  $\lambda, \lambda' \in \tilde{\Lambda}_M$  it holds that

$$\langle \varphi_\lambda, \varphi_{\lambda'} \rangle_{\tilde{F}_M} = c|W|M^n \delta_{\lambda, \lambda'}, \quad (54)$$

where  $c, \varphi_\lambda$  were defined by (1), (41) respectively, and  $n$  is the rank of  $G$ .

**Proof.** The equality

$$\sum_{x \in F_M} \varepsilon(x) \varphi_\lambda(x) \overline{\varphi_{\lambda'}(x)} = \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} \varphi_\lambda(y) \overline{\varphi_{\lambda'}(y)}$$

follows from (10) and (18) and  $W$ -invariance of the expression  $\varphi_\lambda(x) \overline{\varphi_{\lambda'}(x)}$ . Then, using  $W$ -invariance of  $\frac{1}{M} P^\vee / Q^\vee$ , we have

$$\begin{aligned} \langle \varphi_\lambda, \varphi_{\lambda'} \rangle_{F_M} &= \sum_{w' \in W} \sum_{w \in W} \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} (\det ww') e^{2\pi i(w\lambda - w'\lambda', y)} \\ &= |W| \sum_{w' \in W} \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} (\det w') e^{2\pi i(\lambda - w'\lambda', y)} \\ &= c|W|M^n \sum_{w' \in W} (\det w') \delta_{w'\lambda', \lambda}. \end{aligned}$$

If  $\lambda = w'\lambda'$ , then we have from (33) that  $w'\lambda = \lambda = \lambda'$ . Since  $\lambda \in \tilde{\Lambda}_M$ , it follows from (16) and (34) that  $w' = 1$ . □

### 5.4. Discrete C- and S-transforms

Analogously to ordinary Fourier analysis, we define interpolating functions  $\Phi^M, \varphi^M$

$$\Phi^M(x) := \sum_{\lambda \in \Lambda_M} c_\lambda \Phi_\lambda(x), \quad x \in \mathbb{R}^n, \tag{55}$$

$$\varphi^M(x) := \sum_{\lambda \in \tilde{\Lambda}_M} \tilde{c}_\lambda \varphi_\lambda(x), \quad x \in \mathbb{R}^n, \tag{56}$$

which are given in terms of expansion functions  $\Phi_\lambda, \varphi_\lambda$  and expansion coefficients  $c_\lambda$ , whose values need to be determined. These interpolating functions can also be understood as finite cut-offs of infinite expansions. The interpolation properties of (55), (56) were tested in dimensions 2 [5–8] and 3 [9]. In this paper we develop the possibility for solving the interpolation problem for general class of functions sampled on  $F_M$ .

Next we discretize equations (55), (56). Suppose we have some function  $f$  sampled on the grid  $F_M$  or  $\tilde{F}_M$ . The interpolation of  $f$  consists in finding the coefficients  $c_\lambda$  (or  $\tilde{c}_\lambda$ ) in the interpolating functions (55) or (56) such that

$$\Phi^M(x) = f(x), \quad x \in F_M, \tag{57}$$

$$\varphi^M(x) = f(x), \quad x \in \tilde{F}_M. \tag{58}$$

Relations (30), (36) and (52), (54) allow us to view the values  $\Phi_\lambda(x)$  with  $x \in F_M, \lambda \in \Lambda_M$  and the values of  $\varphi_\lambda(x)$  with  $x \in \tilde{F}_M, \lambda \in \tilde{\Lambda}_M$  as elements of non-singular square matrices. These invertible matrices coincide with the matrices of the linear systems (57), (58). Thus, the coefficients  $c_\lambda, \tilde{c}_\lambda$  can be in both cases uniquely determined. The formulas for calculation of  $c_\lambda$  and  $\tilde{c}_\lambda$ , which are also called discrete C- and S-transforms, can be obtained by means of calculation of standard Fourier coefficients:

$$c_\lambda = \frac{\langle f, \Phi_\lambda \rangle_{F_M}}{\langle \Phi_\lambda, \Phi_\lambda \rangle_{F_M}} = (c|W|M^n h_\lambda^\vee)^{-1} \sum_{x \in F_M} \varepsilon(x) f(x) \overline{\Phi_\lambda(x)} \tag{59}$$

$$\tilde{c}_\lambda = \frac{\langle f, \varphi_\lambda \rangle_{\tilde{F}_M}}{\langle \varphi_\lambda, \varphi_\lambda \rangle_{\tilde{F}_M}} = (cM^n)^{-1} \sum_{x \in \tilde{F}_M} f(x) \overline{\varphi_\lambda(x)}. \tag{60}$$

## 6. Concluding remarks

- In situations when a large number of data (functions), sampled on the same grid  $F_M$ , have to be developed into finite series, the processing load can be significantly lightened, which may not be evident from the exposition above. The problem can be viewed in the following matrix form:

$$b = Df, \quad f \in \mathbb{C}^{|\Lambda_M| \times 1}, \quad b \in \mathbb{C}^{|F_M|} \quad D \in \mathbb{C}^{|\Lambda_M| \times |F_M|}.$$

Here  $f$  is a matrix column containing the values of the function sampled on the points of  $F_M$ , which is being decomposed into the finite series of  $|\Lambda_M|$  terms.  $D$  is the decomposition matrix and  $b$  is the matrix column formed by the coefficient of the finite series. Crucial observation here is that  $D$  depends on the values of the C- or S-functions on the points of  $F_M$ , but *not* on the function that is being decomposed. Therefore, the matrix  $D$  needs to be computed only once and then stored as a look-up table for subsequent computations. Moreover, it was shown in [16] that, with some additional constraints, the decomposition matrix can be made to contain only integer matrix elements.

- Products of  $C$ - and  $S$ -functions, referring to the same Lie group and the same point  $x \in F$ , are fully decomposable into the sum of such functions. More precisely,

$$C_\lambda(x)C_{\lambda'}(x) = \sum_k d_k C_k(x), \quad d_k \in \mathbb{N}, \tag{61}$$

$$C_\lambda(x)S_{\lambda'}(x) = \sum_k d_k S_k(x), \quad d_k \in \mathbb{Z}, \tag{62}$$

$$S_\lambda(x)S_{\lambda'}(x) = \sum_k d_k C_k(x), \quad d_k \in \mathbb{Z}. \tag{63}$$

The problem here parallels the decomposition of Weyl group orbits which was extensively studied in [23].

- Reduction of a semisimple Lie group  $G$  to its reductive subgroup, say  $G'$ , implies decomposition of orbit functions of  $G$  into the sum of orbit functions of  $G'$ , so called computation of branching rules. It involves simultaneous transformation of  $\lambda$  and  $x$  to the corresponding quantities of the subgroup.
- Another undoubtedly useful property of the orbit functions, which played no role in this paper, is the fact that they are eigenfunctions of the Laplace operator appropriate for the Lie group and that the eigenvalues are known in all cases.
- There is another type of special function similar to  $C$ - and  $S$ -functions, the so-called  $E$ -functions [1, 9]. Their discretization properties, fundamental domains, orbit sizes and discrete orthogonality relations will be treated in a separated article.
- The present work raises the question under which conditions converge series of the functions  $\{\Phi^M\}_{M=1}^\infty, \{\varphi^M\}_{M=1}^\infty$  assigned to a function  $f : F \rightarrow \mathbb{C}$  by relations (55), (59) and (56), (60).
- The orbit functions are directly related to the symmetric polynomials [24–28], see also section 11 in [2]. A detailed description of these relations as well as their discretization properties deserves further study.

### Acknowledgments

The authors are grateful to R V Moody for fruitful discussions and comments. Work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by the MIND Research Institute, California. JH is grateful for the postdoctoral fellowship and the hospitality extended to him at the Centre de recherches mathématiques, Université de Montréal.

### Appendix

We list the explicit form of the matrices  $R$  for all simple Lie algebras. These matrices determine the number  $|F_M|$  via relation (25):

$A_n, n \geq 1,$

$$R(A_n) = (1)$$

$B_n, n \geq 3,$

$$R(B_n) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$



$C_n, n \geq 2,$

$$R(C_n) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

$D_n, n \geq 4,$

$$R(D_n) = \begin{pmatrix} 1 & 6 & 1 \\ 4 & 4 & 0 \end{pmatrix}$$

$$R(F_4) = \begin{pmatrix} 1 & 84 & 262 & 84 & 1 \\ 1 & 100 & 265 & 66 & 0 \\ 3 & 121 & 253 & 55 & 0 \\ 4 & 139 & 250 & 39 & 0 \\ 8 & 160 & 232 & 32 & 0 \\ 10 & 181 & 220 & 21 & 0 \\ 17 & 199 & 199 & 17 & 0 \\ 21 & 220 & 181 & 10 & 0 \\ 32 & 232 & 160 & 8 & 0 \\ 39 & 250 & 139 & 4 & 0 \\ 55 & 253 & 121 & 3 & 0 \\ 66 & 265 & 100 & 1 & 0 \end{pmatrix} \quad R(G_2) = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 5 & 0 \\ 2 & 4 & 0 \\ 3 & 3 & 0 \\ 4 & 2 & 0 \\ 5 & 1 & 0 \end{pmatrix}$$

$$R(E_6) = \begin{pmatrix} 1 & 132 & 839 & 839 & 132 & 1 \\ 3 & 210 & 945 & 708 & 78 & 0 \\ 9 & 309 & 1017 & 567 & 42 & 0 \\ 20 & 433 & 1038 & 433 & 20 & 0 \\ 42 & 567 & 1017 & 309 & 9 & 0 \\ 78 & 708 & 945 & 210 & 3 & 0 \end{pmatrix}$$

$$R(E_7) = \begin{pmatrix} 1 & 1011 & 19029 & 58150 & 40371 & 5775 & 79 \\ 2 & 1414 & 22208 & 59612 & 36626 & 4510 & 44 \\ 6 & 1944 & 25585 & 60536 & 32856 & 3464 & 25 \\ 12 & 2616 & 29172 & 60816 & 29172 & 2616 & 12 \\ 25 & 3464 & 32856 & 60536 & 25585 & 1944 & 6 \\ 44 & 4510 & 36626 & 59612 & 22208 & 1414 & 2 \\ 79 & 5775 & 40371 & 58150 & 19029 & 1011 & 1 \\ 128 & 7296 & 44032 & 56128 & 16128 & 704 & 0 \\ 208 & 9071 & 47520 & 53654 & 13480 & 483 & 0 \\ 318 & 11142 & 50748 & 50748 & 11142 & 318 & 0 \\ 483 & 13480 & 53654 & 47520 & 9071 & 208 & 0 \\ 704 & 16128 & 56128 & 44032 & 7296 & 128 & 0 \end{pmatrix}$$

$R(E_8) =$

1	1366597	143626378	1624457989	4240905745	3094001119	595138048	20483167	20956
1	1520913	151753498	1668997388	4262344515	3044716089	571669092	18981162	17342
3	1691201	160250557	1714067554	4282411955	2995120797	548866679	17576888	14366
5	1876896	169126731	1759671908	4301058735	2945259720	526733429	16260804	11772
10	2081203	178395851	1805768016	4318297175	2895163955	505251617	15032514	9659
15	2303555	188067805	1852357250	4334079150	2844879535	484421845	13883020	7825
27	2547531	198155859	1899395763	4348420650	2794434993	464226699	12812121	6357
39	2812537	208670852	1946882187	4361275710	2743877271	444664908	11811413	5083
63	3102551	219625367	1994771439	4372663930	2693232237	425719479	10880853	4081
90	3416994	231030959	2043059877	4382541090	2642547860	407387331	10012581	3218
135	3760240	242899625	2091700825	4390930975	2591846890	389652035	9206725	2550
187	4131732	255243637	2140688264	4397791365	2541178068	372508803	8455968	1976
270	4536288	268074057	2189974725	4403149365	2490561450	355941747	7760553	1545
364	4973372	281404062	2239551147	4406965785	2440045746	339944648	7113703	1173
505	5448265	295243652	2289369513	4409270855	2389648355	324502190	6515763	902
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902	6515763	324502190	2389648355	4409270855	2289369513	295243652	5448265	505
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1976	8455968	372508803	2541178068	4397791365	2140688264	255243637	4131732	187
2550	9206725	389652035	2591846890	4390930975	2091700825	242899625	3760240	135
3218	10012581	407387331	2642547860	4382541090	2043059877	231030959	3416994	90
4081	10880853	425719479	2693232237	4372663930	1994771439	219625367	3102551	63
5083	11811413	444664908	2743877271	4361275710	1946882187	208670852	2812537	39
6357	12812121	464226699	2794434993	4348420650	1899395763	198155859	2547531	27
7825	13883020	484421845	2844879535	4334079150	1852357250	188067805	2303555	15
9659	15032514	505251617	2895163955	4318297175	1805768016	178395851	2081203	10
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17342	18981162	571669092	3044716089	4262344515	1668997388	151753498	1520913	1
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25080	22083567	619291548	3142941905	4218087760	1580489625	135855148	1225367	0
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1225367	135855148	1580489625	4218087760	3142941905	619291548	22083567	25080	0

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